# Surface Integrals

$$\iint_S f(x,y,z)\,dS = \iint_D f(x(u,v),\ y(u,v),\ z(u,v))\,|\vec{s}_u \times \vec{s}_v|\,dA$$

where $\vec{s}(u,v)$ parameterizes $S$ on $D$

Ex. Compute $\iint_S x^2$ for the unit sphere centered at the origin.

Sol: First we paremeterize the surface:

$$\vec{S}(\theta, \varphi) = \langle \sin(\varphi)\cos(\theta),\ \sin(\varphi)\sin(\theta),\ \cos\varphi \rangle$$

where $(\theta,\varphi) \in [0,2\pi] \times [0,\pi]$

Sphere where $\rho = 1$, Nicely parameterizes surface

Find $\vec{s}_\theta = \langle -\sin(\varphi)\sin(\theta),\ \sin(\varphi)\cos(\theta),\ 0 \rangle$

$= \sin(\varphi)\langle -\sin\theta,\ \cos\theta,\ 0 \rangle$

$\vec{s}_\varphi = \langle \cos(\varphi)\cos(\theta),\ \cos(\varphi)\sin\theta),\ -\sin(\varphi) \rangle$

$$\vec{s}_\theta \times \vec{s}_\varphi = \sin(\varphi) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \end{vmatrix}$$

VECTOR LAYED OUT HORIZONTALLY

$$\sin\theta \begin{pmatrix} -\sin\varphi\cos\theta \\ -(\sin(\varphi)\sin(\theta)) \\ -\cos(\varphi)\sin\theta^2 - \cos\varphi\cos^2\theta \end{pmatrix}$$

$$= \sin\theta\langle -\sin\varphi\cos\theta,\ -\sin\varphi\sin\theta,\ -\cos\varphi \rangle$$

We popped $\sin(\varphi)$ out of magnitude as $|\sin\varphi|$ is positive on our domain

$$\therefore \iint_S x^2\,dS = \iint_D \sin^3(\varphi)\cos^2(\theta)\,\left|\langle -\sin(\varphi)\cos\theta,\ -\sin(\varphi)\sin(\theta),\ -\cos\varphi\rangle\right| dA$$

$$= \iint_S x^2\,dS = \iint \sin^3(\varphi)\cos^2(\theta)\sqrt{\sin\varphi^2\cos\theta^2 + \sin\varphi^2\sin\theta^2 + \cos\varphi^2}\,dA$$

$$= \iint \sin^3(\varphi)\cos^2(\theta)\,dA$$

$$\int_0^{2\pi} \cos^2(\theta) \int^{\pi} \sin^3(\varphi)\,d\varphi\,d\theta$$

Evaluating inner integral

$$\int_0^{\pi} \sin(\varphi)(1-\cos^2(\varphi))\,d\varphi$$

$u = +\cos\varphi$

$du = -\sin(\varphi)\,d\varphi$

$u(\pi) = -1$

$u(0) = 1$

$$\int_1^{-1} -(1-u^2)\,du$$

$$-\left(u - \frac{1}{3}u^3\right)\Big|_1^{-1}$$

$$-1+\frac{1}{3}-1+\frac{1}{3}$$

$\frac{1}{2}$

$$-\left(\frac{-4}{3}\right) = \frac{4}{3}$$

$$\frac{-4}{3}\int_0^{2\pi} \cos^2(\theta)\,d\theta$$

$V = 2\theta$

$dV = 2\,d\theta$

$V(2\pi) = 4\pi$

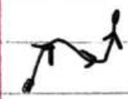
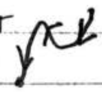
$V(0) = 0$

$$\left(\frac{-4}{3}\right)\frac{1}{2}\int_0^{2\pi}(1+\cos(2\theta))\,d\theta = \frac{-2}{3}\int_0^{4\pi} 1+\cos(v)\,dv = \frac{-2}{3}(2\pi+\sin(2\pi)) = \boxed{\frac{-4}{3}\pi}$$

Goal: Build a theory of surface integrals for vector fields (Analogous to line integrals)

But: We need to think of "orientation" for surfaces first

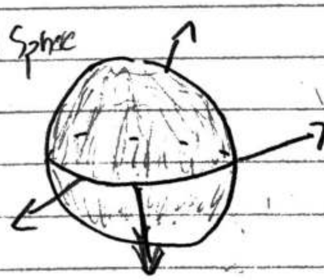
For Line Integrals:

 GOING LEFT vs RIGHT 

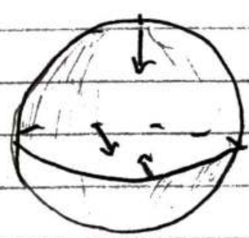
Instead of thinking about orientation, left v Right, think about which way the tangent line points.

Orientation for surfaces means a consistent choice of normal to the tangent

Ex: For A Sphere

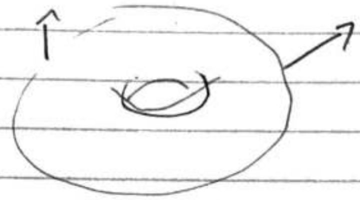


OUTWARD MODEL
OUTWARD ORIENTATION



INWARD ORIENTATION

Alternatively for surfaces that are not closed think "Counter clockwise orientation from above for the tangent plane"



Positive outward orientation

Can we choose a consistent orientation for every surface?

NO

Möbius Strip → has one side, non orientable

Because of such surfaces our theory doesn't work for non-orientable surfaces. From here on out we will work with orientable surfaces.

Note that if/when we choose a parameterization of a surface we automatically chose a orientation. By choosing $\vec{S}(u,v)$ we get a normal:

$$\vec{n}(u,v) = \frac{\vec{S_u} \times \vec{S_v}}{|\vec{S_u} \times \vec{S_v}|}$$

By swapping parameters we swap orientation.

**Defn:** The <u>Flux</u> of v.f. $\vec{v}$ across surface $S$ is $\iint_S \vec{v} \cdot dS = \iint_S \vec{v} \cdot \vec{n}\, dS$

PARAMETRIZE → $= \iint_D \vec{v}(u,v) \cdot \underbrace{\frac{\vec{S}_u \times \vec{S}_v}{|\vec{S}_u \times \vec{S}_v|}}_{\text{SCALAR}} \underbrace{|\vec{S}_u \times \vec{S}_v|}_{\text{SCALAR}}\, dA$

$$= \iint_D \vec{v}(u,v) \cdot (\vec{S}_u \times \vec{S}_v)\, dA$$

where $\vec{S}(u,v)$ is the parametrization of $S$ on domain $D$

**Ex** Compute the flux of $\vec{v}(z,y,x)$ across sphere of radius 1 centered at origin

<u>Convention:</u> If orientation is not given, use outward/positive orientation

**Sol:** As before $\vec{S}(\theta,\varphi) = \begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix}$

$\vec{S}_\theta \times \vec{S}_\varphi = -\sin(\varphi)\begin{pmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{pmatrix}$ (CALCULATED BEFORE)

Is this positive orientation?

At $P = (1,0,0)$ is $\vec{n}$ going out or in, on $S = (0, \pi/2)$

$\vec{n} = -\sin(\frac{\pi}{2})\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ → Pointing in.

Because we found our parameterization to have negative orientation we need to negate our final result

$$\vec{V}(\theta, \varphi)(\vec{S}_\theta \times \vec{S}_\varphi) = \begin{pmatrix} \cos\varphi \\ \sin\varphi \sin\theta \\ \sin\varphi \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \sin\varphi \cos\theta \\ \sin\varphi \sin\theta \\ \cos\varphi \end{pmatrix} (-\sin\varphi)$$

$$= \begin{pmatrix} (\sin\varphi \cos\varphi \cos\theta) + \\ (\sin\varphi^2 \sin\theta^2) + \\ (\sin\varphi \cos\varphi \cos\theta) \end{pmatrix} -\sin\varphi$$

$$= -\sin\varphi\,(2\cos(\varphi)\sin(\varphi)\cos\theta + \sin^2(\varphi)\sin^2(\theta))$$

Negate this because of orientation

$$= \iint_D \sin\varphi\,(2\cos(\varphi)\sin(\varphi)\cos\theta + \sin^2\varphi\,\sin^2(\theta))$$

$$= \underbrace{2\iint_D \sin\varphi^2 \cos\varphi \cos\theta \, dA}_{\displaystyle\int_{\varphi=0}^{\pi} \cos(\varphi)\sin^2(\varphi)\underbrace{\int_0^{2\pi} \cos\theta \, d\theta}_{\text{GOES TO ZERO}}\, d\varphi} + \iint \sin^3\varphi \, \sin^2\theta \, dA$$

$$0 + \iint \sin^3\varphi \, \sin^2\theta \, dA$$